Covering the Aztec Diamond with One-sided Tetrasticks

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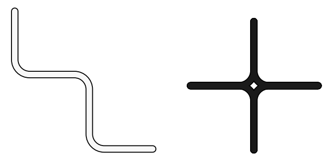
Abstract

There are 107 non-isomorphic coverings of the Aztec diamond of order 5 by the 25 one-sided tetrasticks.

In [6] Donald Knuth describes how to attack certain types of puzzles by computer. One type of puzzle he tried to solve is the covering of patterns by *tetrasticks*. Tetrasticks — or more general *polysticks* — were introduced by Brian Barwell [1]: Pieces made from a fixed number of line segments or sticks. For example, tetrasticks can look like this:

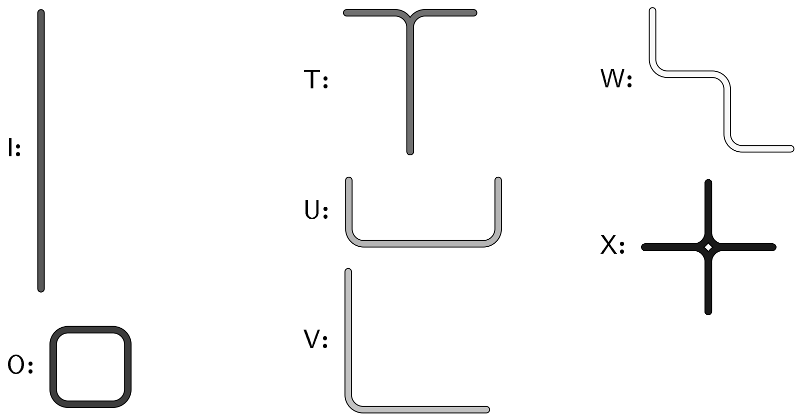


Barwell noted that there are 2 disticks, 5 tristicks, and 16 tetrasticks. With these tetrasticks we try to cover the Aztec diamond pattern.

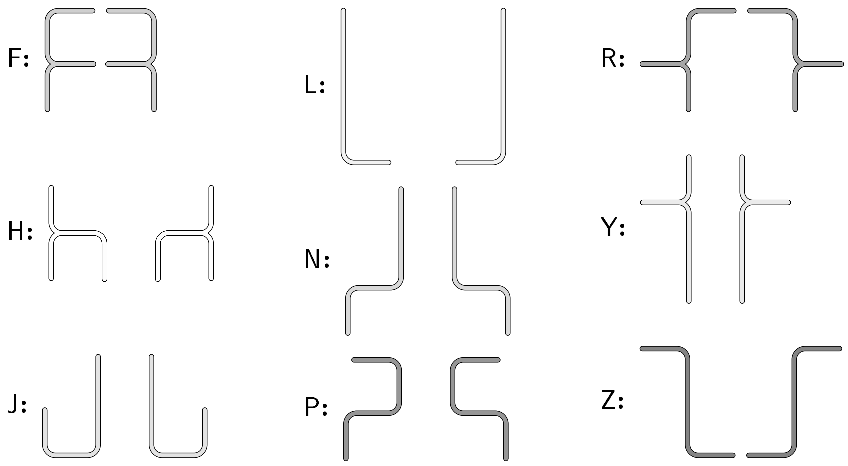
The 25 one-sided tetrasticks

If we add the mirror images of the 9 unsymmetrical pieces to the 16 tetrasticks we get 25 *one-sided* tetrasticks. To get familiar with tetrasticks they are listed in the sequel.

The 7 two-sided tetrasticks. There are 7 tetrasticks with reflective symmetry, named I, O, T, U, V, W and X (the notation differs slightly from Barwells [1]):

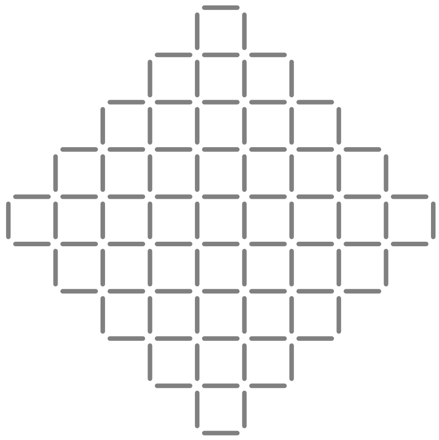


The 9 pairs of one-sided tetrasticks. The 9 pairs of tetrasticks without reflective symmetry are named F, H, J, L, N, P, R, Y and Z:



The Aztec diamond pattern

In [6] the covering of the *Aztec diamond* of order 5 with the 25 *one-sided* tetrasticks is posed as an open problem. An Aztec diamond of order n is the union of unit squares in the plane whose centers are contained in the equation $|x| + |y| \leq n$ [3]:



Donald Knuth writes in [6]: “I also tried unsuccessfully to pack all 25 of the one-sided tetrasticks into the Aztec diamond pattern. . . , but I see no way to prove that a solution is impossible. An exhaustive search seems out of the question at the present time.”

The algorithm

The problem of covering a grid with a fixed set of tetrasticks without crossings of the tetrasticks can be transformed into a Diophantine linear system:

$$\text{Find a } 0/1\text{-vector } x \text{ such that } x \cdot A \leq (1, 1, \ldots, 1), \tag{1}$$

where each line of the 0/1-matrix A corresponds to a possible position of one tetrastick on the underlying grid and the columns of the matrix A consist of

- one column for each tetrastick,
- one column for each line segment of the grid which has to be covered,
- one column for each node of the grid, where an intersection of tetrasticks is possible, i. e. each interior node of the grid.

The first two types of columns have to be fulfilled with equality in (1). The third type can be fulfilled with $\leq$ in (1), it stems from the restriction that tetrasticks are not allowed to cross each other: One interior node of the grid can be covered by at most one straight line formed by two line segments of a tetrasticks.

The problem (1) is known as the *exact cover problem.* In the case of the Aztec diamond pattern the system has $25 + 100 + 40 = 165$ columns and 2945 rows. Knuth gives in [6] a backtracking algorithm to solve the Diophantine linear system (1). The abstract description of the algorithm is quite simple and straight forward:

1. If A is empty, the problem is solved; terminate successfully.
2. Otherwise choose the column c with the least number of nonzero entries.
3. For each row r such that $A[r, c] = 1$
   include r in the partial solution.
   For each j such that $A[r, j] = 1$
      delete column j from the matrix A;
      for each i such that $A[i, j] = 1$,
         delete row i from the matrix A.
4. Repeat this algorithm recursively on the reduced matrix A.

In Step 2 the column c has to be chosen from the first two types of columns in (1), the columns which have to be fulfilled with equality.

The speed of this algorithm is largely determined by the choice of the data structures. Knuth uses in his algorithm DLX doubly linked lists, all the navigation through the matrix is done via pointers. By a trick due to Hitotumatu and Noshita [5], the use of pointers enables a very fast recovering of the original data after stepping back from recursion.

I implemented a parallel version of DLX, based on the software library PVM [4]. This parallel version uses the same strategy as described in [2]: A master process administrates the slave processes which do the backtracking work. As soon as one slave process has finished its part of the backtrack tree, the least progressing slave process splits its backtrack tree into two parts. While this splitting slave process continues to work on one of the two subtrees, on the idle processor the master process starts a new slave process which works on the other part of the splitted backtrack tree.

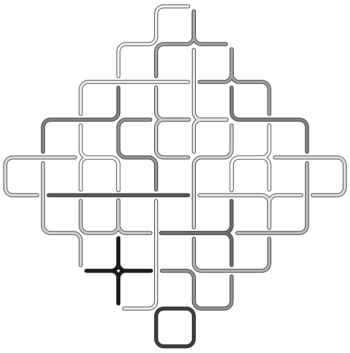
This approach needs after an initial phase of distribution and a final phase of collapse of the search tree very little communication bandwidth. Each machine involved in the search can work on its parts of the search tree without syncronization stops — regardless of its speed.

Coverings of the Aztec diamond pattern

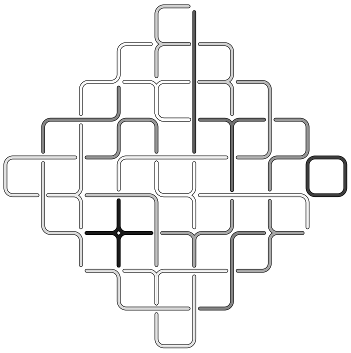
To avoid isomorphic solutions one can restrict the possible positions of one tetrastick. For example, it is enough to try 6 different positions for the tetrastick X. The two positions (2 and 6), where the center of X lies on the diagonal, still leave a reflective symmetry, which can be broken by forcing the tetrastick I to be placed vertical. This system was solved on the 16 node Linux cluster maintained by the Rechenzentrum of the University of Bayreuth: there are exactly 107 non-isomorphic solutions. To measure the amount of work the computers had to do, the number of *updates* have been counted: One update is the removal of one element of the matrix from a doubly linked list. To solve the Aztec diamond challenge the Linux cluster had to perform $13,045,224,997,514$ updates.

Here for each possible position of the tetrastick X the most symmetric solution is shown, i. e. the solution which has the most mirror-image pairs in a symmetric position with respect to an axis of reflection.

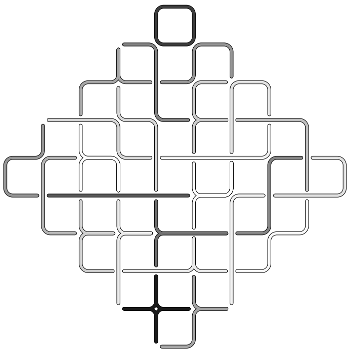
1:



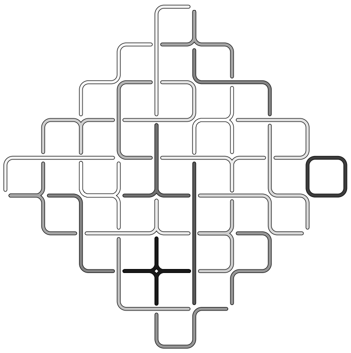
2:



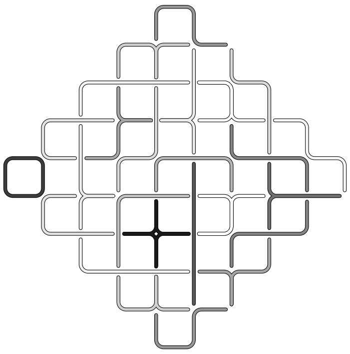
3:



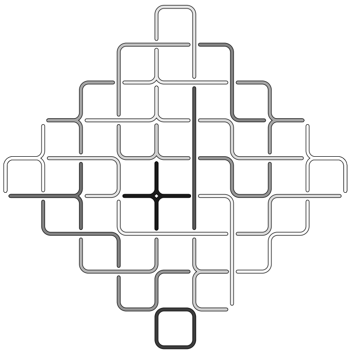
4:



5:



6:



The solutions with the highest symmetry in this respect are the solutions for position 2 and 5, which both have four pairs in symmetric position. No symmetric solutions exist for type 3. The solutions are distributed among the different positions of X according to the following table:

X-position:	1	2	3	4	5	6
number of solutions:	15	11	7	19	36	19

A full list of all solutions can be downloaded from the homepage of the author [7].

Acknowledgments

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References

[1] Brian R. Barwell, "Polysticks," *Journal of Recreational Mathematics* **22** (1990), 165–175.

[2] Anton Betten, Alfred Wassermann, "$\{0,1\}$-Solutions of Integer Linear Equation Systems," in *Parallel Virtual Machine – EuroPVM'96* (Munich, 1996), Lecture Notes in Comput. Sci. 1156, Springer, Berlin, (1996), 311–314.

[3] Noam Elkies, Greg Kuperberg, Michael Larsen, James Propp, "Alternating-Sign Matrices and Domino Tilings," *Journal of Algebraic Combinatorics* **1** (1992), 111–132 and 219–234.

[4] Al Geist, Adam Beguelin, Jack Dongarra, Weicheng Jiang, Robert Manchek, Vaidy Sunderam, *PVM: Parallel Virtual Machine A Users' Guide and Tutorial for Networked Parallel Computing*, MIT Press, Scientific and Engineering Computation, Janusz Kowalik, Editor, (1994) Massachusetts Institute of Technology. The software is downloadable at http://www.epm.ornl.gov/pvm.

[5] Hirosi Hitotumatu, Kohei Noshita, "A technique for implementing backtrack algorithms and its application," *Information Processing Letters* **8** (1979), 174–175.

[6] Donald E. Knuth, "Dancing Links," preprint, (1999), available from `http://www-cs-staff.Stanford.EDU/~knuth/preprints.html`.

[7] Alfred Wassermann, "Covering the Aztec Diamond with One-sided Tetrasticks – Extended Version", available from `http://did.mat.uni-bayreuth.de/wassermann/`.